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Averaged Mappings in the Hilbert Ball*

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Let C be a closed convex subset of a Banach space $(X, |\cdot|)$. Recall that a mapping $T: C \to C$ is said to be nonexpansive if $|Tx - Ty| \leqslant |x - y|$ for all x and y in C. By an averaged mapping $U: C \to C$ we mean a mapping of the form $U = (1-c)I + cT$ where $0 < c < 1$, I is the identity, and $T: C \to C$ is nonexpansive. It is known [1] that averaged mappings have several remarkable properties which are not shared by all nonexpansive mappings. We mention, for example, the following three results.

THEOREM A. *Let C be a closed convex subset of a Banach space X, $U: C \to C$ an averaged mapping, and $d = \inf\{|y - Uy| : y \in C\}$. Then for each x in C, $\lim_{n \to \infty} |U^{n+1}x - U^n x| = d$.*

THEOREM B. *Let C be a closed convex subset of a Banach space X, and let $U: C \to C$ be an averaged mapping with a fixed point. If both X and its dual X^* are uniformly convex, then for each x in C the sequence of iterates $\{U^n x\}$ converges weakly to a fixed point of U.*

THEOREM C. *Let U be an averaged self-mapping of a closed convex subset of a uniformly convex Banach space. Then U is fixed point free if and only if $\lim_{n \to \infty} |U^n x| = \infty$ for all x in C.*

Now let B denote the open unit ball of a complex Hilbert space H, and let $\rho: B \times B \to [0, \infty)$ be the hyperbolic metric on B. For any x and y in B and $0 \leqslant t \leqslant 1$, we let $(1-t)x \oplus ty$ stand for the unique point z in B satisfying $\rho(x, z) = t\rho(x, y)$ and $\rho(z, y) = (1-t)\rho(x, y)$. A mapping $T: B \to B$ is said to be ρ-nonexpansive if $\rho(Tx, Ty) \leqslant \rho(x, y)$ for all x and y in B. Any holomorphic self-mapping of B is ρ-nonexpansive. By an averaged mapping in the Hilbert ball B we mean a mapping $U: B \to B$ of the form $(1-c)I \oplus cT$ where $0 < c < 1$, I is the identity, and $T: B \to B$ is ρ-non-

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expansive. Any such mapping is ρ-nonexpansive. We restrict our attention to self-mappings of B because any ρ-nonexpansive mapping defined on a ρ-closed ρ-convex subset of B can be extended to a ρ-nonexpansive self-mapping of B. In view of the interesting properties of averaged mappings in Banach spaces it is natural to study the class of averaged mappings in the Hilbert ball. This is the main purpose of the present paper.

We begin with an analog of [1, Theorem 2.1].

THEOREM 1. *Let U be an averaged mapping in the Hilbert ball B. Then for each x in B and all $k \geqslant 1$,*

$$\lim_{n \to \infty} \rho(U^{n+1}x, U^n x) = \lim_{n \to \infty} \rho(U^{n+k}x, U^n x)/k = \lim_{n \to \infty} \rho(x, U^n x)/n.$$

Proof. In order to prove the first equality, we fix $x \in B$ and $k \geqslant 1$. Since U is ρ-nonexpansive the limits $L = \lim_{n \to \infty} \rho(U^{n+1}x, U^n x)$ and $R = \lim_{n \to \infty} \rho(U^{n+k}x, U^n x)$ exist. To see that $R \leqslant kL$, we note that $\rho(U^{n+k}x, U^n x) \leqslant \sum_{j=1}^{k} \rho(U^{n+j}x, U^{n+j-1}x) \leqslant k\rho(U^{n+1}x, U^n x)$. To show that $R \geqslant kL$, we recall that $U = (1-c) I \oplus cT$ where $0 < c < 1$ and T is ρ-nonexpansive. Applying [8, Proposition 1], we see that

$$\rho(TU^{n+k}x, U^n x) \geqslant (1-c)^{-k} [\rho(TU^{n+k}x, U^{n+k}x) - \rho(TU^n x, U^n x)] + (1+kc)\,\rho(TU^n x, U^n x).$$

Now we observe that

$$\begin{aligned}\rho(TU^{n+k}x, U^n x) &\leqslant \rho(TU^{n+k}x, TU^n x) + \rho(TU^n x, U^n x)\\ &\leqslant \rho(U^{n+k}x, U^n x) + \rho(TU^n x, U^n x).\end{aligned}$$

Hence

$$\rho(U^{n+k}x, U^n x) \geqslant (1-c)^{-k} [\rho(TU^{n+k}x, U^{n+k}x) - \rho(TU^n x, U^n x)] + (kc)\,\rho(TU^n x, U^n x).$$

Since $\rho(TU^n x, U^n x) = \rho(U^{n+1}x, U^n x)/c$, we see that

$$\rho(U^{n+k}x, U^n x) \geqslant (1-c)^{-k} [\rho(U^{n+k+1}x, U^{n+k}x) - \rho(U^{n+1}x, U^n x)]/c + k\rho(U^{n+1}x, U^n x).$$

Letting $n \to \infty$, we obtain $R \geqslant kL$, as claimed.

To establish the second equality we first note that $\rho(U^{n+k}x, U^n x) \leqslant \rho(U^k x, x)$. Therefore the first equality shows that $\lim_{n \to \infty} \rho(U^{n+1}x, U^n x) \leqslant \rho(U^k x, x)/k$ for all $k \geqslant 1$. Hence $\lim_{n \to \infty} \rho(U^{n+1}x, U^n x) \leqslant \liminf_{k \to \infty} \rho(U^k x, x)/k \leqslant \limsup_{k \to \infty} \rho(U^k x, x)/k \leqslant \limsup_{k \to \infty} \{\sum_{j=1}^{k} \rho(U^j x, U^{j-1}x)\}/$

$k = \lim_{n\to\infty} \rho(U^{n+1}x, U^n x)$. Thus $\lim_{k\to\infty} \rho(U^k x, x)/k$ exists and equals $\lim_{n\to\infty} \rho(U^{n+1}x, U^n x)$. The proof is complete.

Although Theorem 1 does not hold for all ρ-nonexpansive mappings, we remark in passing that $\lim_{n\to\infty} \rho(x, T^n x)/n$ exists for all ρ-nonexpansive $T: B \to B$ (and is independent of x). This fact is a consequence of the following lemma (cf. [10]).

LEMMA 1. *If $\{a(n): n = 1, 2,...\}$ is a sequence of nonnegative real numbers such that $a(m+n) \leqslant a(m) + a(n)$ for all m and n, then $\lim_{n\to\infty} a(n)/n$ exists and equals $\inf\{a(n)/n: n \geqslant 1\}$.*

Proof. Set $L = \inf\{a(n)/n: n \geqslant 1\}$. If $n = km + r$, $0 \leqslant r < m$, then $a(r) \leqslant ra(1)$, $a(km) \leqslant ka(m)$, and $a(n) \leqslant a(r) + a(km) \leqslant ra(1) + ka(m)$. Since $r < m$ and $km \leqslant n$, we have $a(n)/n \leqslant (m/n)\, a(1) + a(m)/m$. Hence

$$L \leqslant \liminf_{n\to\infty} a(n)/n \leqslant \limsup_{n\to\infty} a(n)/n \leqslant a(m)/m.$$

Since we can choose m to make $a(m)/m$ as close as we please to L, we see that $\{a(n)/n\}$ must indeed converge to L.

We now use Theorem 1 to obtain analogs of Theorems A and B.

THEOREM 2. *Let U be an averaged mapping in the Hilbert ball B, and set $d = \inf\{\rho(y, Uy): y \in B\}$. Then for each x in B, $\lim_{n\to\infty} \rho(U^{n+1}x, U^n x) = d$.*

Proof. By Theorem 1, $\lim_{n\to\infty} \rho(U^{n+1}x, U^n x)$ exists and is independent of x. Given a positive ε, there is a point y in B such that $d \leqslant \rho(y, Uy) < d + \varepsilon$. Hence $d \leqslant \lim_{n\to\infty} \rho(U^{n+1}x, U^n x) = \lim_{n\to\infty} \rho(U^{n+1}y, U^n y) \leqslant d + \varepsilon$. The result follows because ε is arbitrary.

Let $\{x_n\}$ be a ρ-bounded sequence in B, and consider the functional $f: B \to [0, \infty)$ defined by $f(x) = \limsup_{n\to\infty} \rho(x_n, x)$. A point z in B is said to be an asymptotic center of the sequence $\{x_n\}$ if $f(z) = \min\{f(x): x \in B\}$. Every ρ-bounded sequence in B has a unique asymptotic center. We shall also need the following fact [7].

LEMMA 2. *If a ρ-bounded sequence $\{x_n\}$ converges weakly to x, then x is the asymptotic center of $\{x_n\}$.*

THEOREM 3. *Let U be an averaged mapping in the Hilbert ball B. If U has a fixed point, then for each x in B the sequence of iterates $\{U^n x\}$ converges weakly to a fixed point of U.*

Proof. Let $x_n = U^n x$, and let a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converge weakly to z. Since U has a fixed point, $\{x_n\}$ is ρ-bounded. Therefore z is the asymptotic center of $\{x_{n_k}\}$ by Lemma 2 and $\lim_{n\to\infty} \rho(x_n, Ux_n) = 0$ by

Theorem 1. It follows that z is a fixed point of U. Hence $\lim_{k\to\infty} \rho(x_{n_k}, z) = \lim_{n\to\infty} \rho(x_n, z)$ and z must be the unique asymptotic center of the whole sequence $\{x_n\}$. Consequently, $\{x_n\}$ converges weakly to z, as claimed.

We do not know if the convergence established in Theorem 3 is actually strong. This is not true in general in the Banach space case [3].

When an averaged mapping $U: B \to B$ is fixed point free, we have more precise information concerning the behavior of its iterates.

Let a belong to the boundary of B, define $\phi_a: B \to (0, \infty)$ by $\phi_a(x) = |1-(x, a)|^2/(1-|x|^2)$, and for positive k consider the ellipsoids $E(a, k) = \{x \in B: \phi_a(x) < k\}$.

We recall [4] that if a ρ-nonexpansive mapping $T: B \to B$ is fixed point free, then there exists a unique point $e = e(T)$ of norm one such that all the ellipsoids $E(e, k)$, $k > 0$, are invariant under T. We shall also use the following lemma.

LEMMA 3. *If a ρ-nonexpansive $T: B \to B$ is fixed point free and* $\lim_{n\to\infty} |T^n x| = 1$ *for some x in B, then the strong* $\lim_{n\to\infty} T^n y = e(T)$ *for all y in B.*

Proof. Denote $T^n x$ by x_n. Since $\phi_e(Ty) \leqslant \phi_e(y)$ for all y in B, $\phi_e(x_n) \leqslant \phi_e(x)$ for all n. Hence $\lim_{n\to\infty}(x_n, e) = 1$ and $\lim_{n\to\infty} x_n = e$. Now consider $y_n = T^n y$. Since $\rho(x_n, y_n) \leqslant \rho(x, y)$ for all n, $\lim_{n\to\infty}(x_n, y_n) = 1$, $\lim_{n\to\infty} |x_n - y_n| = 0$, and the strong $\lim_{n\to\infty} y_n = e(T)$.

THEOREM 4. *Let U be an averaged mapping in the Hilbert ball B. If U is fixed point free, then for each x in B the sequence of iterates $\{U^n x\}$ converges strongly to $e(U)$, a point on the boundary of B.*

Proof. Denote $U^n x$ by x_n and assume that $\{x_n\}$ has a ρ-bounded subsequence $\{x_{n_k}\}$. Then $\lim_{n\to\infty} \rho(x, U^n x)/n = 0$ and $\lim_{n\to\infty} \rho(x_n, Ux_n) = 0$ by Theorem 1. It follows that the asymptotic center of $\{x_{n_k}\}$ is a fixed point of U, a contradiction. Hence $\lim_{n\to\infty} |x_n| = 1$, and the result follows by Lemma 3.

In connection with the difference between Theorems C and 4, it may be of interest to mention an analogy with Brownian motion: In Euclidean space of dimension greater or equal to three, almost all Brownian paths wander out to infinity, but with no asymptotic direction. In hyperbolic space (or more generally, in any complete simply connected Riemannian manifold with curvatures bounded between two negative constants), almost all paths tend to limits on the boundary [11, 14].

The averaged mapping $(1-c)\, I \oplus cT$ is not holomorphic in general, even when T is. Therefore it is natural to consider also mappings of the form $(1-c)\, I + cT$, where $T: B \to B$ is holomorphic, or more generally,

ρ-nonexpansive. We shall call such mappings averaged mappings of the second kind. They are also ρ-nonexpansive. Although we do not have an analog of Theorem 1 for this class of mappings, we are still able to establish, by different methods, analogs of Theorems 3 and 4. This time our arguments are based on the following two facts (cf. [6]).

LEMMA 4. *Let $\{x_n\}$ and $\{z_n\}$ be two sequences in B. Suppose that for some y in B,* $\limsup_{n\to\infty} \rho(x_n, y) \leqslant M$, $\limsup_{n\to\infty} \rho(z_n, y) \leqslant M$, *and* $\liminf_{n\to\infty} \rho((x_n+z_n)/2, y) \geqslant M$. *Then* $\lim_{n\to\infty} |x_n - z_n| = 0$.

Proof. The closed ρ-ball $\bar{B}(y, M)$ is an ellipsoid which consists of all x in B that satisfy $|Px-u|^2/b^2r^2 + |Qx|^2/b^2r \leqslant 1$, where $b = \tanh(M)$, P is the orthogonal projection of H onto the one-dimensional subspace spanned by y, $Q = I - P$, $r = (1-|y|^2)/(1-b^2\,|y|^2)$, and $u = (1-b^2)\,y/(1-b^2\,|y|^2)$. Hence

$$\liminf_{n\to\infty}\{|Px_n - u + Pz_n - u|^2/4b^2r^2 + |Qx_n + Qz_n|^2/4b^2r\} \geqslant 1.$$

Using the parallelogram law, we see that

$$\limsup_{n\to\infty}\{|P(x_n - z_n)|^2/4b^2r^2 + |Q(x_n - z_n)|^2/4b^2r\} \leqslant 0.$$

Therefore $\lim_{n\to\infty} |x_n - z_n| = 0$, as claimed.

A similar proof yields the second fact we need.

LEMMA 5. *Let the point a belong to the boundary of B, and let $\{x_n\}$ and $\{z_n\}$ be two sequences in B. Suppose that* $\limsup_{n\to\infty} \phi_a(x_n) \leqslant M$, $\limsup_{n\to\infty} \phi_a(z_n) \leqslant M$, *and* $\liminf_{n\to\infty} \phi_a((x_n+z_n)/2) \geqslant M$. *Then* $\lim_{n\to\infty} |x_n - z_n| = 0$.

THEOREM 5. *Let V be an averaged mapping of the second kind. If V has a fixed point, then for each x in B the sequence of iterates $\{V^n x\}$ converges weakly to a fixed point of V.*

Proof. Let $V = (1-c)\,I + cT$, where $T\colon B \to B$ is ρ-nonexpansive and $0 < c < 1$. Denote $V^n x$ by x_n, Tx_n by w_n, and let y be a fixed point of T (and V). Assume without loss of generality that $c \leqslant \frac{1}{2}$, and let $z_n = (1-2c)\,x_n + 2cw_n$. Then $x_{n+1} = (x_n+z_n)/2$, $\lim_{n\to\infty} \rho(x_n, y) = M$ exists, $\limsup_{n\to\infty} \rho(w_n, y) \leqslant M$, and $\lim_{n\to\infty} \rho((x_n+z_n)/2, y) = M$. Since any ρ-ball is convex, we also have $\limsup_{n\to\infty} \rho(z_n, y) \leqslant M$. Therefore we can apply Lemma 4 and conclude that $\lim_{n\to\infty} |x_n - z_n| = 0$. Hence $\lim_{n\to\infty} |x_n - w_n| = 0$ too. Since $\{x_n\}$ and $\{w_n\}$ are ρ-bounded, $\lim_{n\to\infty} \rho(x_n, Tx_n) = 0$ and $\{x_n\}$ converges weakly to its asymptotic center, which is a fixed point of T and V.

THEOREM 6. *Let V be an averaged mapping of the second kind. If V is fixed point free, then for each x in B the sequence of iterates $\{V^n x\}$ converges strongly to $e(V)$, a point on the boundary of B.*

Proof. Let $V=(1-c)I+cT$, where $T\colon B\to B$ is ρ-nonexpansive and $0<c<1$. Denote $V^n x$ by x_n, Tx_n by w_n, and let $e=e(T)=e(V)$. Assume without loss of generality that $c\leqslant\frac{1}{2}$, and let $z_n=(1-2c)x_n+2cw_n$. Since $\phi_e(Ty)\leqslant\phi_e(y)$ and $\phi_e(Vy)\leqslant\phi_e(y)$ for all y in B, the sequence $\{\phi_e(x_n)\}$ decreases to a limit M and $\limsup_{n\to\infty}\phi_e(w_n)\leqslant\lim_{n\to\infty}\phi_e(x_n)=M$. The convexity of the ellipsoids $E(e,k)$ now implies that $\limsup_{n\to\infty}\phi_e(z_n)\leqslant M$ too. Since

$$\lim_{n\to\infty}\phi_e((x_n+z_n)/2)=\lim_{n\to\infty}\phi_e(x_{n+1})=M,$$

we can apply Lemma 5 and conclude that $\lim_{n\to\infty}|x_n-z_n|=0$. Hence $\lim_{n\to\infty}|x_n-Tx_n|=0$ too. Since T does not have a fixed point, this implies that $\{x_n\}$ cannot have a ρ-bounded subsequence. Thus $\lim_{n\to\infty}|x_n|=1$ and the result follows from Lemma 3.

There are other classes of nonexpansive and ρ-nonexpansive mappings for which the conclusions of Theorems B, C, 3, and 4 hold. We refer, in particular, to the firmly nonexpansive mappings of the first and second kinds [2, 5, 6]. Without going into details, we recall that a mapping $T\colon C\to C$ is firmly nonexpansive if for each x and y in C, the convex function $f\colon[0,1]\to[0,\infty)$ defined by $f(s)=|(1-s)x+sTx-((1-s)y+sTy)|$ is nonincreasing. We say that a mapping $T\colon B\to B$ is firmly nonexpansive of the first kind if for each x and y in B, the function $g\colon[0,1]\to[0,\infty)$ defined by $g(s)=\rho((1-s)x\oplus sTx,\ (1-s)y\oplus sTy)$ is nonincreasing. Finally, a mapping $T\colon B\to B$ is said to be firmly nonexpansive of the second kind if for each x and y in B, the function $h\colon[0,1]\to[0,\infty)$ defined by

$$h(s)=\rho((1-s)x+sTx,\ (1-s)y+sTy)$$

is nonincreasing.

In certain Banach spaces there is a connection between firmly nonexpansive mappings and averaged mappings. For example, in Hilbert space a mapping is firmly nonexpansive if and only if it is of the form $(I+S)/2$ with a nonexpansive S. This is no longer true in the Hilbert ball (B,ρ). Indeed, although all hyperbolic nearest point projections $R_K\colon B\to K$ onto ρ-closed ρ-convex subsets K of B are firmly nonexpansive of the first kind, not all of them are of the form $\frac{1}{2}I\oplus\frac{1}{2}S$ with a ρ-nonexpansive S [6]. On the positive side, we observe that if an averaged mapping $U=\frac{1}{2}I\oplus\frac{1}{2}T$ has a fixed point y, then the function $\rho((1-s)x\oplus sUx,y)$ decreases on $[0,1]$ for each x in B. To see this, let $2z\ominus x$ denote the unique point u on the metric line passing through x and z which satisfies $z = \frac{1}{2}x \oplus \frac{1}{2}u$, and let $Eq^+(x, y)$ denote the set $\{z \in B: \rho(z, y) \leqslant \rho(z, x)\}$. It is known [6] that the function $\rho((1-s)x \oplus sz, y)$ decreases on $[0, 1]$ if and only if y belongs to $Eq^+(x, 2z \ominus x)$. Since T is ρ-nonexpansive and $2Ux \ominus x = Tx$, $\rho(y, Tx) \leqslant \rho(y, x)$, y belongs to $\mathrm{Eq}^+(x, Tx)$, and the result follows.

We close this paper with an observation concerning Theorem 1. Let C be a closed convex subset of a Banach space X. In addition to the iterates $\{U^n x\}$ of an averaged mapping $U: C \to C$, one could also consider the more general iterative scheme $x_{n+1} = (1 - c_n)x_n + c_n T x_n$, $n \geqslant 0$, where $T: C \to C$ is nonexpansive and $0 \leqslant c_n \leqslant 1$ (see [12], for example).

We claim that if X is uniformly convex and the sequence $\{c_n\}$ is bounded away from 0 and 1, then for each x_0 in C and all $k \geqslant 1$,

$$\lim_{n \to \infty} |x_{n+k} - x_n| \Big/ \sum_{j=n}^{n+k-1} c_j = \lim_{n \to \infty} |x_{n+1} - x_n| / c_n = \lim_{n \to \infty} |x_{n+1}| \Big/ \sum_{j=0}^{n} c_j.$$

To prove the first equality, we combine a generalization of [8, Proposition 1] to the variable coefficients case [9] with the proof of Theorem 1. The second equality follows from the existence of the strong $\lim_{n \to \infty}(x_n - Tx_n)$ [13, Theorem 3.7(c)]. The first equality carries over to the Hilbert ball. We do not know, however, if the second one does too.

Note added in proof. (1) An example of T. Kuczumow and A. Stachura ("Extensions of nonexpansive mappings in the Hilbert ball with the hyperbolic metric," preprint) shows that the convergence established in Theorem 3 is not strong in general. It is based on the example constructed in [3]. (2) The conclusions of Theorems 4 and 6 do not hold for all holomorphic, fixed point free, self-mappings of the (infinite-dimensional) Hilbert ball. In fact, A. Stachura ("Iterates of holomorphic self-maps of the unit ball in Hilbert space," *Proc. Amer. Math. Soc.* **93** (1985), 88–90) has constructed an automorphism $T: B \to B$ for which $0 = \liminf_{n \to \infty} |T^n(0)| < \limsup_{n \to \infty} |T^n(0)| = 1$. (3) Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a holomorphic self-mapping of the open unit disc in the complex plane, and let W be the open unit ball in the space of all bounded linear operators on H. Define a holomorphic $T: W \to W$ by $T(A) = \sum_{n=0}^{\infty} a_n A^n$. Then there exists a complex number w with $|w| \leqslant 1$ such that $\lim_{n \to \infty} T^n(A) = wI$ for all A in W whenever f is fixed point free ($|w| = 1$) or when it has a fixed point but is not an automorphism ($|w| < 1$). In this connection see also the papers by K. Fan, "Iteration of analytic functions of operators," I, II, *Math. Z* **179** (1982), 293–298, and *Linear and Multilinear Algebra* **12** (1983), 295–304.

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